First Order Logic – Lab 2

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First order logic

The lexicon of a first order language contains:

- Connectives & Parentheses: $\neg$, $\rightarrow$, $\leftrightarrow$, $\wedge$, $\vee$, (and);
- Quantifiers: $\forall$ (universal) and $\exists$ (existential);
- Variables: $x, y, z, ...$ ranging over particulars;
- Constants: $a, b, c, ...$ representing a specific element;
- Functions: $f, g, h, ...$, with arguments listed as $f(x_1, ...x_n)$;
- Relations: $R, S, ...$ with an associated arity.

Short-hand notation (optional)

- At least one & at most one: with a "!" after the $\exists$: $\exists! x \phi \leftrightarrow \exists y \forall x (\phi \leftrightarrow x = y)$;
- Numerical restrictions on quantifiers ranging over variable. This can be abbreviated as: $A(y) \rightarrow \exists^{\geq n} x (B(x) \wedge \mathit{relation}(x, y) \wedge n \geq 1)$, or simply to use the number instead of n in the quantification. The general cases for $\exists^{\leq n} x(\phi(x))$ and $\exists^{\geq n} x(\phi(x))$ are:
  - $\exists^{\leq n} x(\phi(x)) \equiv \forall x_1, ..., x_n, x_{n+1}(\phi(x_1) \wedge ... \wedge \phi(x_n) \wedge \phi(x_{n+1}) \rightarrow (x_1 = x_2) \vee ... \vee (x_1 = x_n) \vee (x_1 = x_{n+1}) \vee (x_2 = x_3) \vee ... \vee (x_2 = x_n) \vee (x_2 = x_{n+1}) \vee ... \vee (x_n = x_{n+1}))$
  - $\exists^{\geq n} x(\phi(x)) \equiv \exists x_1, ... x_n(\phi(x_1) \wedge ... \wedge \phi(x_n) \wedge \neg(x_1 = x_2) \wedge ... \wedge \neg(x_1 = x_n) \wedge \neg(x_2 = x_3) \wedge ... \wedge \neg(x_2 = x_n) \wedge ... \wedge \neg(x_{n-1} = x_n))$

First order logic: variable assignments

- Let V be the set of all variables, function $\alpha : V \mapsto \Delta$. Then a *variable assignment* for all free occurrences of x with term d, denoted with $\alpha[x/d]$
- For *closed formulas*, α is omitted, i.e. $\mathcal{I} \models \phi$ (or: $\mathcal{I}$ is a model of ϕ)
- ϕ is satisfied by $\mathcal{I}$ under α, denoted as: $\mathcal{I}, \alpha \models \phi$ (or: interpretation $\mathcal{I}$ is a model of ϕ under α)
- ϕ is valid (a tautology) if every $(\mathcal{I}, \alpha)$ is a model of ϕ, denoted as: $\models \phi$
- ϕ is falsifiable if there is some $(\mathcal{I}, \alpha)$ that does not satisfy ϕ
- Σ entails ϕ, denoted as $\Sigma \models \phi$

Checking

- Consider a first order language where R is a binary relation symbol and P a unary relation symbol (UML class, ER entity type, ORM object type) and an interpretation $\mathcal{I}$ with domain $\{0, 1\}$, where:

$$P^{\mathcal{I}} = \{0, 1\} \quad (1)$$

$$R^{\mathcal{I}} = \{(0, 0), (0, 1)\} \quad (2)$$

- Check whether $\mathcal{I}$ is a model of the following formulas:

$$\forall x \exists y R(x, y) \quad (3)$$

$$\exists x \forall y R(x, y) \quad (4)$$

$$\forall x P(x) \quad (5)$$

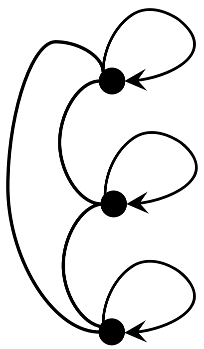
$$\exists x P(x) \quad (6)$$

$$\forall x \forall y (R(x, y) \lor P(x)) \quad (7)$$

$$\forall x \forall y (R(x, y) \land (P(x) \lor P(y))) \quad (8)$$

Another graph

- Find a suitable first-order language and formulate at least two properties of the graph using quantifiers.



Satisfiability under an assignment

- Consider the language $L = \langle R, c \rangle$ where R is a binary relation symbol and c a constant symbol, the L-interpretation $\mathcal{I} = (\mathbb{N}; \cdot^{\mathcal{I}})$ where $R^{\mathcal{I}}$ is the standard linear order $<$ over $\mathbb{N}$, and $c^{\mathcal{I}} = 0$.
- Check whether the following statements hold true. Justify your answers.
  - $\mathcal{I}, \alpha \models \forall x R(c, x)$ for $\alpha(x) = 1$
  - $\mathcal{I}, \alpha \models \exists x R(c, x)$ for $\alpha(x) = 1$
- Do these hold?
  - $\mathcal{I} \models \forall x R(c, x)$
  - $\mathcal{I} \models \exists x R(c, x)$.